

# CONSTRUCTIVE WEAK SOLUTIONS OF THE DIRICHLET PROBLEM

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## ABSTRACT

The paper provides necessary and sufficient conditions for the existence of weak solutions for the Dirichlet problem within Bishop's constructive mathematics. It also considers the continuity of those weak solutions relative to the boundary data and the domain.

### 1. Introduction

Consider the Dirichlet problem

$$\begin{aligned}\Delta u &= 0 && \text{on } \Omega \\ u &= F && \text{on } \partial\Omega\end{aligned}\tag{1}$$

where  $\Omega$  is a totally bounded open subset of  $\mathbb{R}^N$  and  $F$  is a mapping of  $\partial\Omega$  into  $\mathbb{F}$ . (We use the symbol  $\mathbb{F}$  to denote either the real field  $\mathbb{R}$  or the complex field  $\mathbb{C}$ .) Each of the classical approaches to the existence of weak solutions of this problem is unacceptable, as it stands, from a strictly constructive viewpoint. For example, Hilbert's approach, based on Dirichlet's principle, uses the least-upper-bound principle to establish the existence of the infimum of a certain functional and then weak sequential compactness to 'find' the point where that infimum is attained [15, pp. 181–188]; neither the least-upper-bound principle in its usual form nor weak sequential compactness is justifiable within constructive mathematics. On the other hand, an approach based on the Ritz–Galerkin method also makes constructively unacceptable use of the least-upper-bound principle, this time to pass from the boundedness of the sequence of partial sums of a series of nonnegative terms to the convergence of that series [13, pp. 190–195].

Working entirely within the framework of Bishop's constructive mathematics, in this paper we show how a standard classical method, using the representation of linear functionals on a Hilbert space of functions, can be adapted to provide a construction of the weak solution of the Dirichlet problem for a certain type of totally bounded open set  $\Omega \subset \mathbb{R}^N$ . In particular, we show that, if we can solve a certain special case of the Dirichlet problem on  $\Omega$ , we can solve the general case. We then discuss the continuous dependence of the weak solution on the parameters  $F$  and  $\Omega$ .

For background reading on constructive mathematics, we refer the reader to [1, 2, 4, 16]. Note that, since Bishop's constructive mathematics is formally consistent

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with the Church–Markov–Turing thesis, our paper can also be interpreted as a contribution to the recursive theory of partial differential equations; for more information about recursive mathematics, see [11] and [3, Chapter 4].

We assume familiarity with [2, Chapters 3, 6 and 7]. We use the notations of that book, except that we write  $B(a, r)$  for the open ball with centre  $a$  and radius  $r$  in a normed space. Also, we denote by  $\rho$  the metric on any metric space, and by  $\mu$  the Lebesgue measure on  $\mathbb{R}^N$ .

Let  $S$  and  $\Omega$  be subsets of a metric space  $X$ . The *complement* of  $S$  is given by

$$\sim S \equiv \{x \in X : \rho(x, s) > 0 \text{ for all } s \in S\}$$

and the *metric complement* of  $S$  is given by

$$-S \equiv (\sim S)^\circ.$$

We write  $\Omega \sim S$  for  $\Omega \cap (\sim S)$ , and  $\Omega - S$  for  $\Omega \cap (-S)$ .

We say that  $S$  is *well contained* in  $\Omega$  if there exists  $r > 0$  such that  $B(x, r) \subset \Omega$  for each  $x \in S$ ; in this case, we write  $S \subset \subset \Omega$  and we also have  $S \subset \subset \Omega^\circ$ . Classically, every compact subset of an open set  $\Omega \subset \mathbb{R}^N$  is well contained in  $\Omega$ ; this is not the case in recursive constructive mathematics (see [4, Chapter 6]).

We say that  $S$  is *located* (in  $X$ ) if

$$\rho(x, S) \equiv \inf\{\rho(x, s) : s \in S\}$$

exists for each  $x \in X$ ; in this case,

$$-S = \{x \in X : \rho(x, S) > 0\},$$

and  $S \subset \subset \Omega$  if and only if there exists  $r > 0$  such that  $S_r \subset \Omega$ , where

$$S_r \equiv \{x \in X : \rho(x, S) \leq r\}.$$

A bounded subset of  $\mathbb{R}^N$  is located if and only if it is totally bounded.

*We shall assume in the remainder of this paper that  $\Omega$  is a Lebesgue integrable totally bounded open subset of  $\mathbb{R}^N$ , such that  $\partial\Omega$  has Lebesgue measure 0 and Stokes's Theorem applies on  $\bar{\Omega}$ .*

We shall also assume, where necessary and without further comment, that a function defined on  $\Omega$  and having a compact (that is, complete and totally bounded) support  $K$  well contained in  $\Omega$  has been extended continuously to the whole of  $\mathbb{R}^N$  with its value equal to 0 everywhere on  $-K$ ; if the unextended function is  $n$  times uniformly differentiable on  $\Omega$ , then so is its extension to  $\mathbb{R}^N$ . (Assumptions of uniform continuity and uniform differentiability on compact sets are used in Bishop's mathematics since the recursive analogues of the Heine–Borel–Lebesgue Theorem and the uniform continuity theorem are false.)

For each  $n \in \mathbf{N} \cup \{\infty\}$ , let  $\mathcal{C}_0^n(\Omega)$  be the space of functions  $u : \bar{\Omega} \rightarrow \mathbb{F}$  that vanish outside some compact set well contained in  $\Omega$  and that are  $n$ -times uniformly differentiable on  $\Omega$ . The following fundamental result about the space  $\mathcal{C}_0^1(\Omega)$ , the proof of which in [15, pp. 182–183] is essentially constructive, will be of great importance throughout this paper.

LEMMA 1 (Poincaré’s inequality). *There exists a constant  $\gamma > 0$  such that  $\|v\|_2 \leq \gamma \|\nabla v\|_2$  for all  $v \in \mathcal{C}_0^1(\Omega)$ , where  $\|\cdot\|_2$  denotes the  $L^2$  norm.*

Let  $H^1 \equiv H^1(\mathbb{R}^N)$  be the linear space of elements  $u$  of  $L^2(\mathbb{R}^N)$  that are *weakly differentiable*, in the sense that there exist elements  $\partial_1 u, \dots, \partial_N u$  of  $L^2(\mathbb{R}^N)$  (the *weak partial derivatives* of  $u$ ) such that

$$\int_{\Omega} u \partial_k v = - \int_{\Omega} v \partial_k u \quad k = 1, \dots, N$$

for all  $v \in \mathcal{C}_0^1(\mathbb{R}^N)$ ; note that, on the left-hand side of this equation,  $\partial_k$  denotes partial differentiation in the usual sense with respect to the  $k$ th variable. The vector-valued function

$$\nabla u \equiv (\partial_1 u, \dots, \partial_N u)$$

is then called the *weak gradient* of  $u$ .  $H^1$  is complete relative to the norm

$$\|u\|_{H^1} \equiv \|u\|_2 + \|\nabla u\|_2,$$

and

$$\|\nabla u\|_2 = \left( \int_{\Omega} \|\nabla u\|^2 \right)^{1/2} = \left( \int_{\Omega} \sum_{k=1}^N |\partial_k u|^2 \right)^{1/2}.$$

It follows from Poincaré’s inequality that on  $\mathcal{C}_0^1(\Omega)$  the norm associated with the scalar product

$$\langle u, v \rangle \equiv \int_{\Omega} \nabla u \cdot \nabla v^*, \tag{2}$$

where  $v^*$  denotes the complex conjugate of  $v$ , is equivalent to  $\|\cdot\|_{H^1}$ , and therefore that the closure  $H_0^1(\Omega)$  of  $\mathcal{C}_0^1(\Omega)$  with respect to the latter norm is a Hilbert space relative to the scalar product (2). We denote the norm on  $H_0^1(\Omega)$  associated with this scalar product by  $\|\cdot\|_{H_0^1(\Omega)}$  or, when the context is clear, simply by  $\|\cdot\|_H$ . Note that, relative to this norm, the unit ball of  $\mathcal{C}_0^\infty(\Omega)$  (and hence that of  $\mathcal{C}_0^n(\Omega)$  for any positive integer  $n$ ) is dense in  $H_0^1(\Omega)$ ; therefore  $H_0^1(\Omega)$  is separable.

For constructive details of the above results about the spaces  $H^1(\Omega)$  and  $H_0^1(\Omega)$ , see [17].

### 2. The existence of weak solutions

To set the scene for this section, we sketch the classical argument upon which our paper is based. We first rewrite the Dirichlet problem (1) in a form equivalent to the one we started with:

$$\begin{aligned} \Delta u &= f && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3}$$

where this time  $f$  is an  $L^2$  mapping of  $\Omega$  into  $\mathbb{F}$ .

Define a linear functional  ${}_f$  on  $\mathcal{C}_0^1(\Omega)$  as follows:

$${}_f(v) \equiv \int_{\Omega} v f.$$

The inequalities of Hölder and Poincaré show that  ${}_f$  is bounded and therefore that

it extends to a bounded linear functional  $f$  on  $H_0^1(\Omega)$ . Thus there exists a unique element  $u$  of  $H_0^1(\Omega)$  such that

$$f(v) = -\langle v, u^* \rangle \quad (4)$$

for all  $v \in H_0^1(\Omega)$ . An approximation argument, together with Stokes's Theorem, shows that  $u \in H_0^1(\Omega)$  satisfies this condition if and only if

$$\int_{\Omega} u \Delta v = \int_{\Omega} v f \quad (5)$$

for all  $v \in \mathcal{C}_0^\infty(\Omega)$ . An element  $u$  of  $H_0^1(\Omega)$  that satisfies (5) is called a *weak solution* of the Dirichlet problem (3) [10, pp. 117–122].

Although the above method of obtaining a weak solution seems irreproachable, it is not when it is examined constructively: for in order to justify the claim that there exists  $u \in H_0^1(\Omega)$  such that (5) holds, we must show that the linear functional  $f$  is not only bounded but also *normable*, in the sense that

$$\|f\| \equiv \sup \{ |f(v)| : v \in H_0^1(\Omega) \text{ and } \|v\|_H \leq 1 \}$$

exists (can be computed). Indeed, under certain additional conditions, the existence of this norm is equivalent, constructively, to the existence of the infimum of the functional used in Hilbert's justification of the use of Dirichlet's principle to construct weak solutions of the Dirichlet problem (see [17]).

To overcome this difficulty, we first review some basic facts about the constructive counterpart of the weak\* topology on the dual  $H_0^1(\Omega)^*$  of  $H_0^1(\Omega)$ . Let  $(v_n)$  be a dense sequence in  $H_0^1(\Omega)$ . The corresponding *double norm* on  $H_0^1(\Omega)^*$  is defined as follows:

$$\|\lambda\| \equiv \sum_{n=1}^{\infty} 2^{-n} (1 + \|v_n\|_H)^{-1} |\lambda(v_n)|.$$

Double norms arising from different dense sequences in  $H_0^1(\Omega)$  give rise to the same topology on the unit ball

$$B^* \equiv \{ \lambda \in H_0^1(\Omega)^* : \forall v \in H_0^1(\Omega) (|\lambda(v)| \leq \|v\|_H) \}$$

of  $H_0^1(\Omega)^*$ . For each  $u \in H_0^1(\Omega)$ , the mapping  $\lambda \mapsto \lambda(u)$  is uniformly continuous on  $B^*$  (see [2, Chapter 7, (6.3)]). We denote by  $\lambda_v$  the bounded linear functional  $u \mapsto \langle u, v \rangle$  on  $H_0^1(\Omega)$ ; the normable elements of  $H_0^1(\Omega)^*$  are precisely the elements  $\lambda_v$ , and  $\|\lambda_v\| = \|v\|_H$  (see [2, Chapter 8, (2.3)]). If  $S$  is a dense subset of the unit ball of  $H_0^1(\Omega)$ , then the elements  $\lambda_v$  with  $v \in S$  form a dense subset of  $B^*$  (see [2, Chapter 7, (6.7)]) relative to the double norm; so, in particular,

$$S^* \equiv \{ \lambda_v : v \in \mathcal{C}_0^\infty(\Omega), \|v\|_H \leq 1 \}$$

is dense in  $B^*$ .

Recall that the *Fourier transform* of an element  $v$  of  $\mathcal{C}(\mathbb{R}^N)$  is the function  $\hat{v} : \mathbb{R}^N \rightarrow \mathbb{R}$  defined by

$$\hat{v}(\xi) \equiv (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} v(x) dx$$

where  $\xi \cdot x$  denotes the scalar product of  $\xi$  and  $x$  in  $\mathbb{R}^N$ . For a discussion of Fourier transforms, see [15, Chapter 2].

LEMMA 2. *If  $R > 0$  and  $v \in \mathcal{C}_0^1(\Omega)$ , then*

$$\int_{\|\xi\| > R} |\hat{v}(\xi)|^2 d\xi \leq (1 + R^2)^{-1}(\gamma^2 + 1) \|v\|_H^2$$

where  $\gamma$  is the constant in Poincaré's inequality.

*Proof.* Let  $\partial_k$  denote partial differentiation with respect to the  $k$ th variable. Then  $\widehat{\partial_k v}(\xi) = i\xi_k \hat{v}(\xi)$ ; so, by the norm-preserving property of the Fourier transform,

$$\begin{aligned} \int_{\mathbb{R}^N} \xi_k^2 |\hat{v}(\xi)|^2 d\xi &= \int_{\mathbb{R}^N} |\widehat{\partial_k v}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} |\partial_k v(\xi)|^2 d\xi \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^N} \|\xi\|^2 |\hat{v}(\xi)|^2 d\xi &= \sum_{k=1}^N \int_{\mathbb{R}^N} \xi_k^2 |\hat{v}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} \|\nabla v\|^2. \end{aligned}$$

Again using the norm-preserving property of the Fourier transform, we have

$$\begin{aligned} \int_{\|\xi\| > R} |\hat{v}(\xi)|^2 d\xi &= \int_{\|\xi\| > R} (1 + \|\xi\|^2)(1 + \|\xi\|^2)^{-1} |\hat{v}(\xi)|^2 d\xi \\ &\leq (1 + R^2)^{-1} \int_{\mathbb{R}^N} (1 + \|\xi\|^2) |\hat{v}(\xi)|^2 d\xi \\ &= (1 + R^2)^{-1} \left( \int_{\mathbb{R}^N} |\hat{v}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} \|\nabla v\|^2 \right) \\ &= (1 + R^2)^{-1} (\|v\|_2^2 + \|v\|_H^2) \\ &\leq (1 + R^2)^{-1} (\gamma^2 + 1) \|v\|_H^2. \quad \square \end{aligned}$$

Our first main result provides necessary and sufficient conditions for the existence of a weak solution of the Dirichlet problem (3) for a given  $L^2$  function  $f: \Omega \rightarrow \mathbb{F}$ .

THEOREM 1. *The following are equivalent conditions on  $\Omega$ :*

(i) *For each  $\xi \in \mathbb{R}^N$ , the special Dirichlet problem*

$$\begin{aligned} \Delta u &= e^{-i\xi \cdot x} && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{6}$$

*has a weak solution.*

(ii) *For each  $\xi \in \mathbb{R}^N$ , the mapping  $\lambda_v \mapsto \hat{v}(\xi)$  is uniformly continuous on  $S^*$  with respect to the double norm.*

(iii) *The set*

$$S \equiv \{v \in \mathcal{C}_0^1(\Omega) : \|v\|_H \leq 1\}$$

*is totally bounded in the  $L^2$  norm.*

(iv) For each  $L^2$  function  $f: \Omega \rightarrow \mathbb{F}$ , the Dirichlet problem (3) has a weak solution.

*Proof.* Assuming (i), let  $\xi \in \mathbb{R}^N$  and let  $u$  be the weak solution of (6). Then, for each  $v \in \mathcal{C}_0^1(\Omega)$ , we have

$$\begin{aligned} \hat{v}^*(\xi) &= (2\pi)^{-N/2} \int_{\Omega} e^{-i\xi \cdot x} v^*(x) \, dx \\ &= -\langle v^*, (2\pi)^{-N/2} u^* \rangle \\ &= -\langle (2\pi)^{-N/2} u, v \rangle \\ &= -\lambda_v((2\pi)^{-N/2} u). \end{aligned}$$

Statement (ii) now follows since the mapping  $\lambda \mapsto \lambda((2\pi)^{-N/2} u)$  is uniformly continuous with respect to the double norm on  $B^*$ .

Next, note that, by the inequalities of Hölder and Poincaré, for each  $v \in \mathcal{C}_0^1(\Omega)$  and all  $\xi, \xi' \in \mathbb{R}^N$ ,

$$\begin{aligned} |\hat{v}(\xi) - \hat{v}(\xi')| &\leq (2\pi)^{-N/2} \left( \int_{\Omega} |e^{-ix \cdot \xi} - e^{-ix \cdot \xi'}|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |v|^2 \right)^{1/2} \\ &\leq (2\pi)^{-N/2} \gamma \left( \int_{\Omega} |e^{-ix \cdot \xi} - e^{-ix \cdot \xi'}|^2 \, dx \right)^{1/2} \left( \int_{\Omega} \|\nabla v\|^2 \right)^{1/2} \end{aligned}$$

where  $\gamma > 0$  is the constant in Poincaré’s inequality. Now assume (ii), and, given  $\varepsilon > 0$ , choose  $R > 0$  such that

$$(1 + R^2)^{-1}(\gamma^2 + 1) < \frac{\varepsilon^2}{2}$$

and  $\Omega \subset B(0, R)$ . For convenience, set  $B \equiv B(0, 2R)$  and

$$\alpha \equiv \frac{(2\pi)^{N/2} \varepsilon}{2\gamma \sqrt{\mu(\Omega)} \sqrt{2\mu(B)}}.$$

Choose  $t > 0$  such that if  $|z| < t$ , then  $|e^z - 1| < \alpha$ , and let  $\{\xi_1, \dots, \xi_n\}$  be an  $R^{-1}t$ -approximation to the totally bounded set  $B$ . For each  $\xi \in B$ , choose  $k$  such that  $\|\xi - \xi_k\| < R^{-1}t$ . Then

$$|e^{-ix \cdot \xi} - e^{-ix \cdot \xi_k}| = |e^{-ix \cdot (\xi - \xi_k)} - 1| < \alpha$$

for each  $x \in \Omega$ , and so

$$\left( \int_{\Omega} |e^{-ix \cdot \xi} - e^{-ix \cdot \xi_k}|^2 \, dx \right)^{1/2} < \frac{(2\pi)^{N/2} \varepsilon}{2\gamma \sqrt{2\mu(B)}}. \tag{7}$$

By (ii), there exists  $\delta > 0$  such that if

$$v \in \mathcal{C}_0^1(\Omega), \quad \|v\|_H \leq 1, \quad \|\lambda_v\| < \delta \tag{8}$$

then

$$|\hat{v}(\xi_k)| < \frac{\varepsilon}{2\sqrt{2\mu(B)}} \quad 1 \leq k \leq n.$$

Given  $\zeta \in \Omega$ , choose  $k$  such that (7) holds. If  $v$  satisfies (8), then

$$\begin{aligned} |\hat{v}(\zeta)| &\leq |\hat{v}(\zeta) - \hat{v}(\zeta_k)| + |\hat{v}(\zeta_k)| \\ &\leq (2\pi)^{-N/2} \gamma \left( \int_{\Omega} |e^{-ix \cdot \zeta_k} - e^{-ix \cdot \zeta}|^2 dx \right)^{1/2} \left( \int_{\Omega} \|\nabla v\|^2 \right)^{1/2} + \frac{\varepsilon}{2\sqrt{2\mu(B)}} \\ &< (2\pi)^{-N/2} \gamma \frac{(2\pi)^{N/2} \varepsilon}{2\gamma\sqrt{2\mu(B)}} + \frac{\varepsilon}{2\sqrt{2\mu(B)}} \\ &= \frac{\varepsilon}{\sqrt{2\mu(B)}}. \end{aligned}$$

Recalling the norm-preserving property of the Fourier transform and noting Lemma 2, we now see that

$$\begin{aligned} \|v\|_2^2 &= \int_{\mathbb{R}^N} |\hat{v}(\zeta)|^2 d\zeta \\ &\leq \int_B |\hat{v}(\zeta)|^2 d\zeta + \int_{\|\zeta\| > R} |\hat{v}(\zeta)|^2 d\zeta \\ &< \frac{\varepsilon^2}{2\mu(B)} \mu(B) + (1 + R^2)^{-1} (\gamma^2 + 1) \|v\|_H^2 \\ &\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} \\ &= \varepsilon^2. \end{aligned}$$

Thus, relative to the double norm on  $S^*$ ,  $\lambda_v \mapsto v$  is a uniformly continuous mapping into  $L^2$ . However,  $S^*$  is dense in  $B^*$ , and is therefore totally bounded, relative to the double norm, so the set

$$S \equiv \{v \in \mathcal{C}_0^1(\Omega) : \|v\|_H \leq 1\}$$

is totally bounded in the  $L^2$  norm. Thus (ii) implies (iii).

Since, as the inequality

$$\left| \int_{\Omega} vf \right| \leq \left( \int_{\Omega} v^2 \right)^{1/2} \left( \int_{\Omega} |f|^2 \right)^{1/2}$$

shows, the mapping  $v \mapsto \int_{\Omega} vf$  is uniformly continuous with respect to the  $L^2$  norm on  $\mathcal{C}_0^1(\Omega)$ , it follows that if (iii) holds, then

$$\sup \left\{ \left| \int_{\Omega} vf \right| : v \in \mathcal{C}_0^1(\Omega), \|v\|_H \leq 1 \right\}$$

exists. In this case, as the unit ball of the subspace  $\mathcal{C}_0^1(\Omega)$  is dense in that of  $H_0^1(\Omega)$ , the linear functional  $v \mapsto \int_{\Omega} vf$  on  $\mathcal{C}_0^1(\Omega)$  extends to a normable linear functional on  $H_0^1(\Omega)$ . Applying the Riesz Representation Theorem [2, Chapter 8, (2.3)], we then obtain  $u \in H_0^1(\Omega)$  such that

$$\int_{\Omega} vf = -\langle v, u^* \rangle \quad v \in H_0^1(\Omega).$$

Hence  $u$  is a weak solution of (3), and so (iii) implies (iv). The final implication, (iv)  $\Rightarrow$  (i), is trivial.  $\square$

Property (iii) of Theorem 1 holds classically as a special case of the Rellich Compactness Theorem [15, Theorem 1, p. 202].

In principle, we can extract from our proof of Theorem 1 an algorithm which would enable us to convert the data for weak solutions of the special Dirichlet problem (6), for arbitrary  $\zeta \in \mathbb{R}^N$ , into a weak solution of the general Dirichlet problem (3) (see [14] and [1, Chapter XI]). In practice, the extraction and encoding of this algorithm would be a highly non-trivial business.

### 3. Continuity in parameters of weak solutions

Experience shows that constructive proofs of the existence of classically unique objects embody information about their continuity in parameters, from which uniqueness follows immediately. In this section, we discuss the continuous dependence of the weak solutions of the Dirichlet problem (3) on the parameters  $f$  and  $\Omega$ .

In view of the equivalence of the forms (1) and (3) of the Dirichlet problem, we can regard the following result as an expression of the continuity of weak solutions with respect to the boundary data.

**THEOREM 2.** *For each  $L^2$  function  $f: \Omega \rightarrow \mathbb{F}$ , denote by  $u_f$  the weak solution of the Dirichlet problem (3). Then for all  $L^2$  functions  $f, g: \Omega \rightarrow \mathbb{F}$ ,*

$$\|u_f - u_g\|_H \leq \gamma \|f - g\|_2$$

where  $\gamma$  is the constant in Poincaré's inequality.

*Proof.* An approximation argument shows that

$$\int_{\Omega} v(f - g) = -\langle v, (u_f - u_g)^* \rangle$$

for all  $v \in H_0^1(\Omega)$ . Taking  $v = u_f - u_g$  and using Poincaré's inequality, we obtain

$$\begin{aligned} \|u_f - u_g\|_H^2 &= \left| \int_{\Omega} (u_f - u_g)(f - g) \right| \\ &\leq \|u_f - u_g\|_2 \|f - g\|_2 \\ &\leq \gamma \|u_f - u_g\|_H \|f - g\|_2 \end{aligned}$$

from which the desired inequality follows.  $\square$

In order to discuss the continuity of weak solutions relative to the parameter  $\Omega$ , we require four lemmas, the first three of which deal with located sets in  $\mathbb{R}^N$ .

For points  $x, y \in \mathbb{R}^N$ , we write

$$\begin{aligned} (x, y) &\equiv \{tx + (1-t)y : 0 < t < 1\} \\ [x, y] &\equiv \{tx + (1-t)y : 0 \leq t \leq 1\}. \end{aligned}$$

Note that, if  $x \neq y$  (that is,  $|x - y| > 0$ ), then  $[x, y]$  is closed in  $\mathbb{R}$ .

**LEMMA 3.** *Let  $U$  be a subset of  $\mathbb{R}^N$  such that  $U \cup \sim U$  is dense, and let  $x_0 \in U$ ,  $y_0 \in \sim U$ , and  $\varepsilon > 0$ . Then there exists  $z \in \partial U$  such that  $\rho(z, [x_0, y_0]) < \varepsilon$ .*

*Proof.* See [6].  $\square$

We measure the closeness of totally bounded subsets  $A, B$  of  $\mathbb{R}^N$  by the *Hausdorff distance*

$$\rho(A, B) \equiv \max \{m(A, B), m(B, A)\}$$

where

$$m(A, B) \equiv \sup \{\rho(x, B) : x \in A\}.$$

LEMMA 4. *Let  $S, S'$  be totally bounded subsets of  $\mathbb{R}^N$  with totally bounded boundaries, and let  $\delta$  be a positive number such that*

$$\max \{\rho(S, S'), \rho(\partial S, \partial S')\} < \delta.$$

Then

$$\rho(x, \sim S) = \rho(x, \partial S) \leq 2\delta$$

for each  $x \in \bar{S} \sim S'$ .

*Proof.* Given  $x \in \bar{S} \sim S'$  and  $\varepsilon > 0$ , choose  $x' \in S'$  such that  $\|x - x'\| < \rho(x, S') + \varepsilon$ . Since  $S'$ , and therefore  $\bar{S}'$ , is located,  $\bar{S}' \cup \sim \bar{S}'$  is dense in  $\mathbb{R}^N$ ; it follows from Lemma 3 that there exists  $z \in \partial S'$  such that  $\rho(z, [x, x']) < \varepsilon$ . Then

$$\begin{aligned} \rho(x, \partial S) &\leq \rho(x, \partial S') + \delta \\ &\leq \|x - z\| + \delta \\ &\leq \|x - x'\| + \rho(z, [x, x']) + \delta \\ &< \rho(x, S') + \varepsilon + \varepsilon + \delta \\ &\leq 2\delta + 2\varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we conclude that  $\rho(x, \partial S) \leq 2\delta$ . □

LEMMA 5. *If  $\Omega$  is a subset of a Banach space such that  $\Omega \cup \sim \Omega$  is dense and  $\partial \Omega$  is located, then  $\Omega$  and  $\sim \Omega$  are located, and  $\rho(x, \sim \Omega) = \rho(x, \partial \Omega)$  for each  $x \in \bar{\Omega}$ .*

*Proof.* See [6]. □

Our last preliminary result is the Cutoff Lemma, the proof of which in [12, p. 202] can easily be modified for our constructive purposes (see also [5]).

LEMMA 6. *There exists a constant  $\kappa$ , depending only on  $N$ , such that, for each compact set  $K \subset \mathbb{R}^N$  and each  $r > 0$ , there exists a  $C^\infty$  function  $\alpha : \mathbb{R}^N \rightarrow [0, 1]$  with the following properties:*

- (i)  $\alpha(x) = 1$  for all  $x \in K$ .
- (ii)  $\alpha(x) = 0$  whenever  $\rho(x, K) \geq r$ .
- (iii)  $\|\nabla \alpha(x)\| \leq \kappa/r$  for all  $x \in \mathbb{R}^N$ .

For convenience, we call an open subset  $\Omega$  of  $\mathbb{R}^N$  *admissible* if the following properties hold:

- (i)  $\Omega$  is totally bounded and Lebesgue integrable.
- (ii)  $\partial \Omega$  is totally bounded and has Lebesgue measure 0.
- (iii) The Dirichlet problem (3) has a weak solution for each  $L^2$  function  $f : \Omega \rightarrow \mathbb{F}$ .

Note that, if  $\Omega$  is admissible, then  $\Omega \cup \sim \Omega$  is dense, and so, by Lemma 5,  $\sim \Omega$  is located. Also, we could have weakened the total boundedness of  $\Omega$  to mere

boundedness in property (i) since, as is shown in [6], if a bounded, Lebesgue measurable subset of  $\mathbb{R}^N$  has located boundary, then the subset itself is located and therefore totally bounded.

We call a subset  $S$  of a metric space *coherent* if  $-(\sim S) \subset S$ . Coherence and related properties are discussed at length in [6].

With these definitions, we are equipped, at last, to deal with the continuous dependence of weak solutions on the domain of the Dirichlet problem.

**THEOREM 3.** *Let  $\Omega$  be an admissible open subset of  $\mathbb{R}^N$ , let  $f$  be an  $L^2$  function from  $\Omega$  into  $\mathbb{F}$ , and let  $u$  be the weak solution of the corresponding Dirichlet problem (3). Assume also that the following are true:*

(i) *For each  $\varepsilon > 0$ , there exists an integrable compact set  $K \subset\subset \Omega$  such that  $\mu(\Omega - K) < \varepsilon$ .*

(ii) *There exists a constant  $c_0 > 0$  such that, if*

$$\partial\Omega_r \equiv \{x \in \mathbb{R}^N : \rho(x, \partial\Omega) \leq r\}$$

*is integrable, then*

$$\int_{\partial\Omega_r} |u|^2 \leq c_0 r^2 \int_{\partial\Omega_r} \|\nabla u\|^2$$

*for all  $u \in H_0^1(\bar{\Omega})$ .*

*Then, for each  $\varepsilon > 0$ , there exists  $\delta > 0$  with the following property.*

*If  $\Omega'$  is a coherent admissible open set well contained in  $\Omega$  such that*

$$\max\{\rho(\Omega, \Omega'), \rho(\partial\Omega, \partial\Omega')\} < \delta,$$

*then  $\|u - u'\|_{H_0^1(\Omega)} < \varepsilon$ , where  $u'$  is the weak solution of the Dirichlet problem*

$$\begin{aligned} \Delta u' &= f && \text{on } \Omega' \\ u' &= 0 && \text{on } \partial\Omega' \end{aligned}$$

*and  $u'$  is extended to equal 0 outside  $\Omega'$ .*

*Proof.* Let  $\kappa$  be as in Lemma 6, and let  $\varepsilon > 0$ . Choose  $t > 0$  such that, if  $S$  is an integrable set and  $\mu(\Omega - S) < t$ , then

$$\int_{\Omega - S} \|\nabla u\|^2 < \frac{2\varepsilon^2}{9(1 + 25c_0\kappa^2)}.$$

By hypothesis (i), there exists a compact integrable set  $S \subset\subset \Omega$  such that  $\mu(\Omega - S) < t$ . Choose  $\delta$  such that

$$0 < \delta < \frac{1}{5} \inf\{\|x - y\| : x \in S, y \in \partial\Omega\},$$

and

$$K \equiv \{x \in \bar{\Omega} : \rho(x, \sim \Omega) \geq 5\delta\}$$

is compact and integrable [2, Chapter 6, (6.3)]. Clearly, if  $\rho(x, K) < 5\delta$ , then  $\rho(x, \Omega) = 0$ ; since

$$\{x \in \mathbb{R}^N : \rho(x, K) < 5\delta\}$$

is dense in  $K_{5\delta}$ , it follows that  $K_{5\delta} \subset \bar{\Omega}$ . On the other hand, using Lemma 3, we can easily show that if  $x \in S$ , then  $\rho(x, K) = 0$ ; thus  $S \subset \bar{K} = K$ , and therefore  $\mu(\Omega - K) < t$ .

Let  $\Omega' \subset \subset \Omega$  be a coherent admissible open set such that

$$\max\{\rho(\Omega', \Omega), \rho(\partial\Omega', \partial\Omega)\} < \delta.$$

We show that  $K_\delta \subset \subset \Omega'$ . To this end, consider  $x \in K_{2\delta}$ , and note that  $B(x, \delta) \subset K_{3\delta} \subset \Omega$ . If there exists  $y \in B(x, \delta) \sim \Omega'$ , then  $\rho(y, \sim \Omega) \leq 2\delta$ , by Lemma 4, and therefore  $\rho(x, \sim \Omega) < 3\delta$ ; choosing  $z \in K$  such that

$$\|x - z\| < 2\delta + (3\delta - \rho(x, \sim \Omega)),$$

we find that

$$\begin{aligned} \rho(z, \sim \Omega) &\leq \|x - z\| + \rho(x, \sim \Omega) \\ &< 5\delta, \end{aligned}$$

contradicting our definition of  $K$ . It follows that  $\rho(x, \sim \Omega') \geq \delta$ ,  $x \in -(\sim \Omega')$ , and therefore, by the coherence of  $\Omega'$ ,  $x \in \Omega'$ . Thus  $K_{2\delta} \subset \Omega'$ , and therefore  $K_\delta \subset \subset \Omega'$ .

Using Lemma 6, now construct a  $C^\infty$  function  $\alpha: \mathbb{R} \rightarrow [0, 1]$  such that  $\alpha(x) = 0$  if  $x \in K$ ,  $\alpha(x) = 1$  if  $x \in -K_\delta$ , and  $\|\nabla\alpha(x)\| \leq \kappa/\delta$  for all  $x \in \mathbb{R}^N$ . Let  $u' \in H_0^1(\Omega')$  be the weak solution of the Dirichlet problem

$$\begin{aligned} \Delta u' &= f \quad \text{on } \Omega' \\ u' &= 0 \quad \text{on } \partial\Omega'. \end{aligned}$$

Approximation arguments show that  $\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega v^* f$  for all  $v \in H_0^1(\Omega)$ , that  $\int_{\Omega'} \nabla u' \cdot \nabla v = -\int_{\Omega'} v^* f$  for all  $v \in H_0^1(\Omega')$ , and hence that

$$\int_{\Omega'} (\nabla u - \nabla u') \cdot \nabla v = 0 \quad v \in H_0^1(\Omega').$$

Now define a function  $w \equiv u - u'$ . Let  $(u_n)$  be a sequence in  $\mathcal{C}_0^1(\Omega)$  converging to  $u$  in  $\|\cdot\|_{H_0^1(\Omega)}$ , and let  $(u'_n)$  be a sequence in  $\mathcal{C}_0^1(\Omega')$  converging to  $u'$  in  $\|\cdot\|_{H_0^1(\Omega')}$ . Then  $(1 - \alpha)(u_n - u'_n)$  belongs to  $\mathcal{C}_0^1(\Omega')$ , and so

$$\int_{\Omega'} \nabla w \cdot \nabla((1 - \alpha)u_n - u'_n) = 0.$$

Therefore, in the limit as  $n$  tends to  $\infty$ ,

$$\int_{\Omega'} \nabla w \cdot \nabla(w - \alpha u) = 0.$$

Noting that

$$2ab \leq sa^2 + \frac{b^2}{s} \quad a, b \in \mathbb{R}, s > 0,$$

we now obtain

$$\begin{aligned} 2 \int_{\Omega'} \|\nabla w\|^2 &= 2 \int_{\Omega'} \alpha \nabla w \cdot \nabla u + 2 \int_{\Omega'} u \nabla w \cdot \nabla \alpha \\ &\leq \frac{1}{3} \int_{\Omega'} \|\nabla w\|^2 + 3 \int_{\Omega'} \alpha^2 \|\nabla u\|^2 \\ &\quad + \frac{1}{3} \int_{\Omega'} \|\nabla w\|^2 + 3 \int_{\Omega'} \|\nabla \alpha\|^2 |u|^2. \end{aligned}$$

It follows from this and our choice of  $\alpha$  that

$$4 \int_{\Omega'} \|\nabla w\|^2 \leq 9 \int_{\Omega-K} \|\nabla u\|^2 + \frac{9\kappa^2}{\delta^2} \int_{\Omega-K} |u|^2.$$

Now yet another application of Lemma 3 shows that  $\bar{\Omega} - K = \bar{\Omega} \cap \partial\Omega_{5\delta}$ . Since  $\mu(\partial\Omega) = 0$  and  $u = 0$  on  $-\Omega$ , we see from hypothesis (ii) that

$$\begin{aligned} \int_{\Omega-K} |u|^2 &= \int_{\partial\Omega_{5\delta}} |u|^2 \\ &\leq c_0(5\delta)^2 \int_{\partial\Omega_{5\delta}} \|\nabla u\|^2 \\ &= 25c_0\delta^2 \int_{\Omega-K} \|\nabla u\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} 4 \int_{\Omega'} \|\nabla u - \nabla u'\|^2 &= 4 \int_{\Omega'} \|\nabla w\|^2 \\ &\leq 9(1 + 25c_0\kappa^2) \int_{\Omega-K} \|\nabla u\|^2. \end{aligned}$$

Since  $\mu(\Omega - K) < t$ , it follows that

$$\int_{\Omega'} \|\nabla u - \nabla u'\|^2 < \frac{\varepsilon^2}{2}.$$

Since  $K \subset \Omega'$ , we also have  $\mu(\Omega - \Omega') < t$ , so that

$$\begin{aligned} \int_{\Omega} \|\nabla u - \nabla u'\|^2 &= \int_{\Omega - \Omega'} \|\nabla u\|^2 + \int_{\Omega'} \|\nabla u - \nabla u'\|^2 \\ &< \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} \\ &= \varepsilon^2 \end{aligned}$$

and therefore  $\|u - u'\|_{H^1_b(\Omega)} \leq \varepsilon$ . □

Hypothesis (i) of Theorem 3 is discussed in [6]. Hypothesis (ii) of Theorem 3, and property (ii) of an admissible open set, hold classically when  $\partial\Omega$  is a smooth embedded compact manifold of dimension  $N - 1$  (see [15, Chapter 5, Theorem 6]); since there is, as yet, no well developed constructive theory of manifolds, we prefer to adopt these hypotheses rather than follow diverting paths that lead into manifold theory.

As far as we know, Theorem 3 is new even in classical mathematics. The only related classical results that we have found are one dealing with the dependence of Green's functions on the domain [7, p. 291], and a consequent one on the continuity of strong solutions of the Dirichlet problem when the domain satisfies certain strong uniform boundary conditions [8].

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