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The anti-Specker property, positivity, and total boundedness

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Working within Bishop-style constructive mathematics, we examine some of the consequences of the anti-Specker property, known to be equivalent to a version of Brouwer's fan theorem. The work is a contribution to constructive reverse mathematics.

In this paper we continue the discussion [4, 8], within Bishop's constructive mathematics (**BISH**)[1)], of generalisations of the anti-Specker property for $[0,1]$ (relative to $\mathbf{R}$) – that is,

> If $(z_n)_{n\geqslant 1}$ is a sequence in $\mathbf{R}$ that is eventually bounded away from each point of $[0,1]$, then $(z_n)_{n\geqslant 1}$ is eventually bounded away from the entire interval $[0,1]$.[2)]

Specifically, we are interested in

– the connection between generalised anti-Specker properties and the positivity of the infimum of a positive-valued function on a compact – that is, complete, totally bounded – metric space, and

– the question whether every space with a generalised anti-Specker property is totally bounded.

This work lies within the programme of constructive reverse mathematics, in which, on the one hand, we examine the constructive equivalence of classical statements (see, for example, [11, 17]), and on the other, we seek, for example, to classify theorems according to the version of the fan theorem to which they are equivalent [14, 15, 16, 26]. It is known that, within **BISH**, the anti-Specker property for the interval $[0,1]$ is equivalent to Brouwer's fan theorem $\mathbf{FT}_c$ for 'c-bars'.[3)] Although $\mathbf{FT}_c$ is not adopted as a principle of **BISH**, since it holds in the intuitionistic model of **BISH** it (and therefore the anti-Specker property for compact metric spaces) can be regarded as more-or-less constructive, at least provided you are prepared to dispense with a recursive interpretation of your constructive mathematics.

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1) This is simply mathematics carried out with intuitionistic logic and within some suitable set-theoretic framework such as that found in [1]. We shall also allow the use of countable and dependent choice.

2) The anti-Specker property is in direct opposition to Specker's theorem, a fundamental result in recursive analysis [25]. For more about the anti-Specker property, see [7, 8].

3) There are currently four versions of Brouwer's fan theorem that have been investigated in the scope of constructive reverse mathematics. All of them enable one to conclude that a given bar is uniform; the difference between them lies in the required complexity of the bar. This ranges from the very strongest requirement – decidable – to no restriction on the bar at all. Between these two extremes lie fan theorems for bars that are c-sets and Π^0_1-sets, respectively. These two fan theorems, $\mathbf{FT}_c$ and $\mathbf{FT}_{\Pi^0_1}$, are of particular interest, since one can show that the proof-theoretic strength of the uniform continuity theorem for continuous functions on compact metric spaces lies between them; but whether that theorem is actually equivalent to either $\mathbf{FT}_c$ or $\mathbf{FT}_{\Pi^0_1}$ remains an open question. Many other analytical theorems have, however, been shown to be equivalent to versions of the fan theorem. As it is more convenient in constructive analysis to work with purely analytical principles, rather than logical ones, the anti-Specker property for the interval $[0,1]$ is an important principle to investigate. For more on these matters, see [3, 4].

In order to arrive at a good generalisation of the anti-Specker property for $[0,1]$, we need some definitions and results from [7].

Let X be a subspace of a metric space Z, and $\zeta \in Z$. For convenience, we write '$\rho(\zeta, X) > 0$' to signify that there exists $c > 0$ such that $\rho(\zeta, x) \geqslant c$ for each $x \in X$, and '$r \geqslant \rho(\zeta, X)$' to signify that $r \geqslant \rho(\zeta, x)$ for each $x \in X$. Note that in doing so, we make no assumption that the distance $\rho(\zeta, X)$ exists as an infimum.[4] We define the *metric complement* of X in Z to be the set

$$Z - X \equiv \{z \in Z : \rho(z, X) > 0\}.$$

If X is a subspace of Z such that $Z - X$ is inhabited (that is, contains a point), then we call Z a *metric superspace* of X. By a *one-point extension* of X we mean a metric superspace Z of X such that $Z - X$ consists of a single point. It is straightforward to construct one-point extensions of a given metric space (X, ρ).

We recall that a sequence $(x_n)_{n \geqslant 1}$ in a metric space (Z, ρ) is said to be

– *eventually bounded away from the point* $x \in Z$ if there exist N and $\delta > 0$ such that $\rho(x, x_n) > \delta$ for all $n \geqslant N$;

– *eventually bounded away from the subset* X of Z if there exist N and $\delta > 0$ such that $\rho(x, x_n) > \delta$ for all $x \in X$ and all $n \geqslant N$;

– *eventually not in* X if there exists N such that $x_n \notin X$ for all $n \geqslant N$;

– *detachable from* X, or X-*detachable*, if for each n, either $x_n \in X$ or $x_n \notin X$.

The following two results were proved in [7, Propositions 1,2].

Proposition 1 *The following are equivalent conditions on a metric space* X.

(i) *There exists a metric superspace* Z *of* X *such that every* X*-detachable sequence in* Z *that is eventually bounded away from each point of* X *is eventually bounded away from* X.

(ii) *For every metric superspace* Z *of* X*, every* X*-detachable sequence in* Z *that is eventually bounded away from each point of* X *is eventually not in* X.

(iii) *For every one-point extension* Z *of* X*, every sequence in* Z *that is eventually bounded away from each point of* X *is eventually not in* X.

(iv) *There exists a one-point extension* Z *of* X *such that every sequence in* Z *that is eventually bounded away from each point of* X *is eventually not in* X.

(v) *There exists a metric superspace* Z *of* X *such that every sequence in* Z *that is eventually bounded away from each point of* X *is eventually bounded away from* X.

Proposition 2 *The following are equivalent conditions on the metric subspace* $[0,1]$ *of* $\mathbf{R}$.

(a) *Every sequence in* $\mathbf{R}$ *that is eventually bounded away from each point of* $[0,1]$ *is eventually bounded away from* $[0,1]$.

(b) *Every* $[0,1]$*-detachable sequence in* $\mathbf{R}$ *that is eventually bounded away from each point of* $[0,1]$ *is eventually bounded away from* $[0,1]$.

(c) *Every sequence in* $\mathbf{R}$ *that is eventually bounded away from each point of* $[0,1]$ *is eventually not in* $[0,1]$.

In view of these results, we call the following the (*unrelativised*) *anti-Specker property*[5] of the metric space X:

$\mathbf{AS}^1_X$ For some one-point extension Z of X, every sequence in Z that is eventually bounded away from each point of X is eventually not in X.

If this property holds for some one-point extension of X, then, by Proposition 1, it holds for every one-point extension of X. With this definition at hand, we now turn to the main work of the paper.

[4] In fact, $\rho(\xi, X)$ is what Richman [23] calls an *upper real.*

[5] A "relativised" version of the anti-Specker property is introduced in [7] along with $\mathbf{AS}^1_X$.

Arguably the first paper in the modern era of constructive reverse mathematics is that of Julian and Richman [18], in which it is shown that a version of the fan theorem, $\mathbf{FT}_{\mathrm{D}}$, weaker than $\mathbf{FT}_{\mathrm{c}}$, is equivalent to the *positivity property*

$\mathbf{POS}_X$ Every positive-valued, uniformly continuous function f on X has positive infimum

for $X = [0,1]$.[6] Our first lemma will enable us to derive two extensions of the implication from a fan theorem to $\mathbf{POS}_X$; in these extensions, we weaken the continuity requirement to a pointwise one, at the cost of replacing $\mathbf{FT}_{\mathrm{D}}$ by the anti-Specker property. For the proof of this lemma we note that a mapping $f : X \longrightarrow Y$ between metric spaces is *nonconstant* if there exist $x, x' \in X$ such that $f(x) \neq f(x')$ in Y.

Lemma 3 *Let $f : X \longrightarrow Y$ be a mapping between metric spaces, and let $x_0 \in X$. If f is pointwise continuous at x_0, then it has the following property:*

(*) *For each sequence $(x_n)_{n\geqslant 1}$ in X, if $(f(x_n))_{n\geqslant 1}$ is eventually bounded away from $f(x_0)$ in Y, then $(x_n)_{n\geqslant 1}$ is eventually bounded away from x_0 in X.*

Conversely, if (*) holds and X is complete, then f is sequentially continuous at x_0. Furthermore, if (*) holds, f is sequentially continuous at x_0, and X is separable, then f is pointwise continuous at x_0.

Proof. It is easy to show that if f is pointwise continuous, then (*) holds. Conversely, suppose that f has the property (*), and let the sequence $(x_n)_{n\geqslant 1}$ converge to x_0 in X. Now, (*) implies that f is strongly extensional, by Ishihara's tricks [13, Lemmas 1, 2]; so for each $\varepsilon > 0$ we have either $\rho(f(x_0), f(x_n)) < \varepsilon$ for all sufficiently large n or else $\rho(f(x_0), f(x_n)) > \varepsilon/2$ infinitely often; but in the latter case, it follows from (*) that the sequence $(x_n)_{n\geqslant 1}$ has a subsequence that is bounded away from x_0, which is absurd. Hence f is sequentially continuous.

Now let $(a_n)_{n\geqslant 1}$ be a dense sequence in X such that

$$\forall_n \forall_k \exists_m (m > k \wedge a_m = a_n). \tag{1}$$

Assume first that f is nonconstant, so there exists $\xi \in X$ with $f(\xi) \neq f(x_0)$. Given $\varepsilon > 0$, construct a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \Rightarrow \rho(f(x_0), f(a_n)) > \frac{\varepsilon}{2}, \quad \text{and} \quad \lambda_n = 1 \Rightarrow \rho(f(x_0), f(a_n)) < \varepsilon.$$

If $\lambda_n = 0$, set $z_n \equiv a_n$; if $\lambda_n = 1$, set $z_n \equiv \xi$. For each n we have

$$\rho(f(z_n), f(x_0)) > \min\left\{\frac{\varepsilon}{2}, \rho(f(x_0), f(\xi))\right\}.$$

It follows from (*) that there exist $\delta > 0$ and N such that $\rho(z_n, x_0) > \delta$ for all $n \geqslant N$. Given $x \in X$ such that $\rho(x_0, x) < \delta$, suppose that $\rho(f(x_0), f(x)) > \varepsilon$. By the sequential continuity of f and (1), there exists $n > N$ such that $\rho(x_0, a_n) < \delta$ and $\rho(f(x_0), f(a_n)) > \varepsilon$; then $\lambda_n = 0$, $z_n = a_n$, and therefore $\rho(z_n, x_0) < \delta$. Since this contradicts our choice of N, we conclude that $\rho(f(x_0), f(x)) \leqslant \varepsilon$. Hence f is pointwise continuous at x_0.

To remove the nonconstancy condition on f, first let $Z \equiv X \cup \{\zeta\}$ be a one-point extension of X, and extend f to Z by setting $f(\xi) \equiv f(x_0)$; replacing X by Z, we may assume that X contains a point $\xi \neq x_0$. Next, define a mapping F of X into the product metric space $Y \times \mathbf{R}$ by

$$F(x) \equiv (f(x), \rho(x, x_0)).$$

Then F is nonconstant. We show that it also satisfies the appropriate version of (*). Let $(x_n)_{n\geqslant 1}$ be a sequence in X such that there exist $\delta > 0$ and N with

$$\max\{\rho(f(x_n), f(x)), \rho(x_n, x_0)\} = \rho(F(x_n), F(x_0)) > \delta$$

for all $n \geqslant N$; we may assume that $N = 1$. Construct a binary sequence $(\alpha_n)_{n\geqslant 1}$ such that

$$\alpha_n = 0 \Rightarrow \rho(f(x_n), f(x_0)) > \delta, \quad \text{and} \quad \alpha_n = 1 \Rightarrow \rho(x_n, x_0) > \delta.$$

[6] The status of $\mathbf{POS}_X$ in the context of point-free topology is dealt with in [22].

If $\alpha_n = 0$, set $y_n \equiv x_n$; if $\alpha_n = 1$, set $y_n \equiv \xi$. Then

$$\rho\left(f(y_n), f(x_0)\right) \geqslant \min\{\delta, \rho\left(f(\xi), f(x_0)\right)\}.$$

It follows from (*) applied to f that the sequence $(y_n)_{n\geqslant 1}$ is eventually bounded away from x_0. Pick $\gamma > 0$ and ν such that $\rho\left(y_n, x_0\right) \geqslant \gamma$ for all $n \geqslant \nu$. Then $\rho\left(x_n, x_0\right) \geqslant \min\{\delta, \gamma\}$ for all such n. This completes the proof that (*) holds with f replaced by F and with Y replaced by $Y \times \mathbf{R}$. It follows that F, and therefore f, is pointwise continuous at x_0. □

The first of our results about $\mathbf{POS}_X$ is rather weak.

Proposition 4 *Let X be a metric space that has the anti-Specker property, and let $f : X \longrightarrow \mathbf{R}$ be a pointwise continuous, positive-valued function. Then* $\neg(\forall \varepsilon > 0)(\exists x \in X)\left(f(x) < \varepsilon\right)$.

Proof. Let Z be a one-point extension of X. Suppose that for each $\varepsilon > 0$ there exists $x \in X$ with $f(x) < \varepsilon$. Then there exists a sequence $(x_n)_{n\geqslant 1}$ in X such that $f(x_n) < 1/n$ for each n. Let $x \in X$, and compute N such that $f(x) > 2/N$. Then $|f(x) - f(x_n)| > 1/N$ for all $n \geqslant N$. It follows from Lemma 3 that the sequence $(x_n)_{n\geqslant 1}$ is eventually bounded away from x. Since $x \in X$ is arbitrary, the anti-Specker property of X forces that sequence to be eventually not in X. This is clearly absurd. □

Proposition 5 *Let X be a metric space with the anti-Specker property, and let $f : X \longrightarrow \mathbf{R}$ be a pointwise continuous, positive-valued function whose infimum exists. Then* $\inf f > 0$.

Proof. Construct a one-point extension $Z \equiv X \cup \{\zeta\}$ of X, where $\rho\left(\zeta, X\right) > 0$. Then construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf f < 1/n, \quad \text{and} \quad \lambda_n = 1 \Rightarrow \inf f > 1/\left(n+1\right).$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, choose $z_n \in X$ such that $f(z_n) < 1/n$. If $\lambda_n = 1$, set $z_n \equiv \zeta$. Given $x \in X$, we show that $(z_n)_{n\geqslant 1}$ is eventually bounded away from x. To do so, first compute a positive integer such that $f(x) > 2/N$. If $\lambda_N = 1$, then for each $n \geqslant N$, $\rho\left(x, z_n\right) \geqslant \rho\left(\zeta, X\right)$. We may therefore assume that $\lambda_N = 0$. For each $k \geqslant N$, define $n_k \equiv k$ if $\lambda_k = 0$, and $n_k \equiv n_{k-1}$ if $\lambda_k = 1$. If $\lambda_k = 0$, then

$$|f(x) - f(z_k)| \geqslant f(x) - f(z_k) > \frac{2}{N} - \frac{1}{n_k} \geqslant \frac{1}{N}.$$

Thus the sequence $(f(z_{n_k}))_{k\geqslant N}$ is eventually bounded away from $f(x)$. It follows from Lemma 3 that there exist $\delta > 0$ and $K \geqslant N$ such that $\rho\left(x, z_{n_k}\right) > \delta$ for all $k \geqslant K$. Hence $\rho\left(x, z_n\right) \geqslant \delta$ for all $n \geqslant N$ such that $\lambda_n = 0$. We conclude that $\rho\left(x, z_n\right) \geqslant \min\{1, \delta\}$ for all $n \geqslant N$. This completes the proof that $(z_n)_{n\geqslant 1}$ is eventually bounded away from each $x \in X$. It follows that the sequence is eventually not in X; so $z_n = \zeta$, and therefore $\lambda_n = 1$, for some n. Hence $\inf f > 0$. □

If X has the anti-Specker property, does every positive-valued, continuous function f on X have an infimum? What if f is uniformly continuous? To answer this, we note that if a metric space X has the property

"every uniformly continuous, positive-valued mapping on X has a positive lower bound",

then X is totally bounded [6].

Proposition 6 *If X is a separable metric space with the anti-Specker property, then the following are equivalent:*

(i) *X is totally bounded.*

(ii) *Each uniformly continuous, positive-valued mapping on X has an infimum.*

Proof. It is well known that (i) implies (ii); see [10, Corollary 2.2.7]. Conversely, assume (ii). Then, by Proposition 5, every uniformly continuous, positive-valued function on X has a positive infimum. It follows from the theorem in [6], quoted above, that X is totally bounded. □

The preceding proposition leads to the question: is every separable metric space with the anti-Specker property totally bounded? A partial answer is given by the next three results, though none of them has total boundedness as a consequence.

Proposition 7 *Let* X *be a metric space with the anti-Specker property, let* $(x_n)_{n\geqslant 1}$ *be a located sequence in* X*, and let* $\varepsilon > 0$*. Then it is impossible that* $\rho(x_m, x_n) > \varepsilon$ *for all distinct* m *and* n*.*

Proof. Suppose that $\rho(x_m, x_n) > \varepsilon$ for all distinct m and n. Given $x \in X$, we have

$$\text{either} \quad \rho(x, \{x_n : n \geqslant 1\}) > \varepsilon/4 \quad \text{or} \quad \rho(x, \{x_n : n \geqslant 1\}) < \varepsilon/2.$$

In the latter case, pick N such that $\rho(x, x_N) < \varepsilon/2$; then $\rho(x, x_n) > \varepsilon/2$ for all $m \neq N$. Thus the sequence $(x_n)_{n\geqslant 1}$ is eventually bounded away from each point of X. By the anti-Specker property, it is eventually not in X, which is absurd. □

The following strengthens Proposition 7.

Proposition 8 *Let* X *be a separable metric space with the anti-Specker property, let* $(x_n)_{n\geqslant 1}$ *be a located sequence in* X*, and let* $\varepsilon > 0$*. Then for all sufficiently large* n *there exists* $m < n$ *such that* $\rho(x_m, x_n) < \varepsilon$*.*

Proof. Construct a one-point extension $Z \equiv X \cup \{\zeta\}$ of X, where $\rho(\zeta, X) > 0$. Given $\varepsilon > 0$, construct a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that $\lambda_1 = 0$ and, for $n \geqslant 2$,

$$\lambda_n = 0 \Rightarrow (\forall k < n)(\rho(x_n, x_k) > \varepsilon/2), \quad \text{and} \quad \lambda_n = 1 \Rightarrow (\exists k < n)(\rho(x_n, x_k) < \varepsilon).$$

If $\lambda_n = 0$, set $z_n \equiv x_n$; if $\lambda_n = 1$, set $z_n \equiv \zeta$. Since $(x_n)_{n\geqslant 1}$ is located, for a given x in X either $\rho(x, x_n) > \varepsilon/8$ for all n, or else there exists N such that $\rho(x, x_N) < \varepsilon/4$. In the latter case, consider any $n > N$. If $\lambda_n = 0$, then $z_n = x_n$ and $\rho(x_N, x_n) > \varepsilon/2$, so $\rho(x, z_n) > \varepsilon/4$. If $\lambda_n = 1$, then $z_n = \zeta$ and so $\rho(x, z_n) \geqslant \rho(\zeta, X)$. We now see that the sequence $(z_n)_{n\geqslant 1}$ is eventually bounded away from each $x \in X$. Hence, by Proposition 1, there exists ν such that $z_n \notin X$ for all $n \geqslant \nu$. For all such n we must have $\lambda_n = 1$; whence there exists $m < n$ with $\rho(x_n, x_m) < \varepsilon$. □

Corollary 9 *Let* X *be a separable metric space with the anti-Specker property, let* $(x_n)_{n\geqslant 1}$ *be a dense sequence in* X*, and let* $\varepsilon > 0$*. Then for all sufficiently large* n *there exists* $m < n$ *such that* $\rho(x_m, x_n) < \varepsilon$*.*

Proof. This follows from Proposition 8, since dense sequences are located. □

We return shortly to the total boundedness question, for which we need

Proposition 10 *Let* X *be a metric space with the anti-Specker property, and let* f *be a pointwise continuous mapping of* X *onto a metric space* Y*. Then* Y *has the anti-Specker property relative to each of its one-point extensions.*

Proof. Construct a one-point extension $Z \equiv X \cup \{\zeta\}$ of X, with $\rho(\zeta, X) > 0$, and a one-point extension $W \equiv Y \cup \{\omega\}$ of Y, with $\rho(\omega, Y) > 0$. Defining $f(\zeta) \equiv \omega$, we extend f to a pointwise continuous mapping of Z onto W. Let $(z_n)_{n\geqslant 1}$ be a sequence in Z such that $(f(z_n))_{n\geqslant 1}$ is eventually bounded away from each point of X. By Lemma 3, $(x_n)_{n\geqslant 1}$ is eventually bounded away from each point of X; whence, by the anti-Specker property, there exists N such that $x_n = \zeta$, and therefore $f(x_n) = \omega$, for all $n \geqslant N$. It follows that $\rho(f(x_n), Y) = \rho(\omega, Y) > 0$ for all $n \geqslant N$. Thus $(f(x_n))_{n\geqslant 1}$ is eventually bounded away from Y. □

Recall that a metric space X is *pseudocompact* if every pointwise continuous mapping of X into **R** is bounded. It is shown in [5, Theorem 2] that a separable pseudocompact metric space is totally bounded. Note, though, that in the recursive interpretation of **BISH**, the interval $[0, 1]$ is totally bounded but not pseudocompact [2, Section IV.7, Theorem (ii)].

Theorem 11 *In* **BISH***, the following are equivalent:*

(i) *Every separable metric space with the anti-Specker property is pseudocompact.*

(ii) *Every separable metric space with the anti-Specker property is totally bounded.*

P r o o f. The implication from (i) to (ii) is a special case of [5, Theorem 2]. Conversely, assume (ii) and let X be a separable metric space with the anti-Specker property. Given a continuous mapping $f : X \longrightarrow \mathbf{R}$, we see from Proposition 10 that $f(X)$ has the anti-Specker property. Since $f(X)$ is clearly separable, it follows from (ii) that it is totally bounded. Hence f is bounded. □

To give a more complete answer to the question posed before Proposition 7, we recall a definition and a principle, both due to Ishihara. First the definition: a set A of real numbers is called *pseudobounded* if for each sequence $(a_n)_{n\geqslant 1}$ in A, $a_n/n \to 0$ as $n \to \infty$. Now

BD-N Every inhabited, countable, pseudobounded set of positive integers is bounded.

Note that, as is easily shown using countable choice, we may replace *positive integers* by *real numbers* in this last statement. The principle **BD-N** holds in the three standard models of **BISH**: classical mathematics (essentially, **BISH** plus the law of excluded middle), Brouwer's intuitionistic mathematics (**BISH** plus Brouwer's continuity principle and fan theorem), and recursive constructive mathematics (**BISH** plus the Church-Markov-Turing thesis) [14]. However, it is not derivable in a certain formal version of **BISH** [19]. For more on **BD-N**, see [24].

Proposition 12 *In* **BISH + BD-N**, *every separable metric space with the anti-Specker property is pseudocompact.*

P r o o f. Given a separable metric space X with the anti-Specker property, construct a one-point extension $Z \equiv X \cup \{\zeta\}$ of X, where $\rho(\zeta, X) > 0$. Let $(x_n)_{n\geqslant 1}$ be a dense sequence in X, and $f : X \to \mathbf{R}$ a pointwise continuous mapping. We show that the inhabited countable subset

$$A \equiv \{|f(x_n)| : n \geqslant 1\}$$

of $\mathbf{R}$ is pseudobounded. Given an increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers, construct a binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\lambda_k = 0 \Rightarrow |f(x_{n_k})| > k - 1, \quad \text{and} \quad \lambda_k = 1 \Rightarrow |f(x_{n_k})| < k.$$

We may assume without loss of generality that $\lambda_1 = 0$. If $\lambda_k = 0$, set $z_k \equiv x_{n_k}$; if $\lambda_k = 1$, set $z_k \equiv \zeta$. Consider any $x \in X$. Since f is continuous at x, there exist $\delta > 0$ and a positive integer K such that if $x' \in X$ and $\rho(x, x') < \delta$, then $|f(x')| < K - 1$. If $k > K$ and $\rho(x_{n_k}, x) < \delta$, then $\lambda_k = 1$ and therefore $\rho(z_k, x) \geqslant \rho(\zeta, X)$. Thus $(z_k)_{k\geqslant 1}$ is eventually bounded away from each point of X, and hence eventually not in X. We can therefore compute κ such that for each $k \geqslant \kappa$, $\lambda_k = 1$ and therefore $|f(x_{n_k})| < k$. It follows from [24, Theorem 1.1] that A is pseudobounded. By **BD-N**, it is bounded; whence, by continuity, the function f is bounded. □

Corollary 13 *In* **BISH + BD-N**, *every separable metric space with the anti-Specker property is totally bounded and has the positivity property* $\mathbf{POS}_X$.

P r o o f. The total boundedness follows from Proposition 12 and Theorem 11. Propositions 6 and 5 then show that the space has the property $\mathbf{POS}_X$. □

Can we drop **BD-N** from Proposition 12? Suppose we could do so. Then the c-fan theorem $\mathbf{FT}_c$, equivalent (over **BISH**) to the anti-Specker property for $[0, 1]$, would imply that $[0, 1]$ is pseudocompact; in turn, this would entail the uniform continuity theorem:

UCT Every pointwise continuous mapping from a compact metric space into a metric space is uniformly continuous.

(See [9, 20].) Our experience with fan theorems suggests to us that the weakest version of the fan theorem that implies **UCT** is likely to be stronger than $\mathbf{FT}_c$, and hence that we cannot drop **BD-N** from Proposition 12.

So much for total boundedness. What about the (classically valid) implication from $\mathbf{AS}_X$ to the completeness of X? To deal with this, we recall the essentially nonconstructive principle :

LPO For each binary sequence $(a_n)_{n\geq 1}$, either $a_n = 0$ for all n or else there exists n with $a_n = 1$.

Proposition 14 *If every subspace of* $[0, 1]$ *with the anti-Specker property is complete, then* **LPO** *holds.*

Proof. Consider the subspace

$$X \equiv \{0\} \cup \left\{ \frac{1}{n} : n \geqslant 1 \right\}$$

of $[0, 1]$. Let $(x_n)_{n \geqslant 1}$ be a sequence in **R** that is eventually bounded away from each point of X. Then there exists a positive integer N such that $|x_n| > 1/N$ for each $n \geqslant N$. Pick a positive integer $\nu > N$ such that $\left|x_n - \frac{1}{k}\right| > \frac{1}{\nu}$ for all $n \geqslant \nu$ and $k \in \{1, \ldots, N\}$. Then $|x_n - x| > \frac{1}{\nu}$ for all $x \in X$ and all $n \geqslant \nu$. Thus $(x_n)_{n \geqslant 1}$ is eventually bounded away from X. Hence X has the anti-Specker property. Given a binary sequence $(a_n)_{n \geqslant 1}$, define a (clearly Cauchy) sequence $(\xi_n)_{n \geqslant 1}$ in X such that if $a_n = 0$, then $\xi_n = 1/n$, and if $a_n = 1 - a_{n-1}$, then $\xi_k = \xi_{n-1}$ for all $k \geqslant n$. If X is complete, then this sequence converges to a limit $\xi \in X$. If $\xi = 0$, then $a_n = 0$ for all n. If $\xi = 1/N$ for some N, then $a_N = 1$. □

Does **LPO** imply the completeness of every subspace of $[0, 1]$ with the anti-Specker property? It would be enough to prove, with the aid of **LPO**, that such a space X is closed in $[0, 1]$; but to do so seems highly unlikely without more information about the criterion for membership of X. In fact, the proof-theoretic strength of the statement

"every subspace of $[0, 1]$ with the anti-Specker property is complete"

seems to lie strictly between **LEM** and **LPO**, and therefore in unexplored territory in constructive reverse mathematics.

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