A Weak Constructive Sequential Compactness Property and the Fan Theorem

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Abstract

A weak constructive sequential compactness property of metric spaces is introduced. It is proved that for complete, totally bounded metric spaces this property is equivalent to Brouwer's fan theorem for detachable bars. Our results form a part of constructive reverse mathematics.

Keywords: constructive, sequential compactness, fan theorem, reverse mathematics We investigate situations of the following general type within Bishop's constructive framework **BISH**—that is, using intuitionistic logic and an appropriate set–theoretic foundation such as Aczel's CST [1]. We have a certain property P, applicable to elements of a set X with an inequality $\neq$, and some strong form of negation of P, which we denote by P'; generally this negation will be stronger than the standard negation of P in constructive logic. We also have a classical proof that there exists $x \in X$ with the property P. It has long been appreciated[1] that if classically there is a *unique* $x \in X$ with the property P, then there is a reasonable chance that, perhaps with the help of some extra constructive principles, we can produce a constructive proof of the existence of that x. There is now a considerable body of classical proof–theoretic work, by Kohlenbach and others [11, 12, 16], showing how classical unique existence proofs can be converted into constructive existence proofs. However, in the work below *we discuss constructive results entirely within constructive metamathematics*—that is, using intuitionistic logic and making no reference to classical theories; see also [13]. This approach is analogous to that of classical recursion theory, in which only recursion–theoretic methods are used to prove theorems, such as Specker's theorem [14], some of which are almost obvious when viewed from outside the recursive framework. In order to introduce the idea of uniqueness without *a priori* existence, let us say that there is **at most one element of** X **with the property** P if

$$\forall_{x,y \in X} \left(x \neq y \Rightarrow P'(x) \vee P'(y) \right).$$

We are interested in the connection between the statement

$$\forall_{x,y \in X} \left(x \neq y \Rightarrow P'(x) \vee P'(y) \right) \Rightarrow \exists_{x \in X} \ P(x) \tag{0.1}$$

—if there is at most one element of X with the property P, then there is such an element—and Brouwer's fan theorem. The examination of such connections is a part of constructive

[1] According to the very helpful referee's report on this paper, the connection between classical unique existence and constructive existence can be traced back to the work of Lacombe in the 1950s.

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reverse mathematics [9]. Berger et al. have shown that (0.1) is equivalent to the fan theorem in the following two cases.

1. X is a compact metric space, $f : X \to \mathbb{R}$ is a uniformly continuous mapping, $P(x)$ is the statement $f(x) = \inf_{y \in X} f(y)$, and $P'(x)$ is the statement $f(x) > \inf_{y \in X} f(y)$. [3]
2. X is a compact metric space; f is a uniformly continuous mapping of X into itself that has approximate fixed points, in the sense that

$$\forall_{\varepsilon>0} \exists_{x \in X} \left(\rho(x, f(x)) < \varepsilon\right);$$

$P(x)$ is the statement $f(x) = x$; and $P'(x)$ is the statement $\rho\left(x, f(x)\right) > 0$. [2]

In this note we discuss a special case of the general problem that deals with a constructive notion of sequential compactness. Let X be a metric space. We say that a sequence $\mathbf{x} = (x_n)_{n \geqslant 0}$ in X has **at most one cluster point,**[2] if for all distinct x, y in X, there exist $\delta > 0$ and a positive integer N such that

$$\forall_{n \geqslant N} \left(\rho\left(x, x_n\right) \geqslant \delta\right) \vee \forall_{n \geqslant N} \left(\rho(y, x_n) \geqslant \delta\right). \tag{0.2}$$

We say that X is **sequentially compact** if every sequence in X that has at most one cluster point actually has a cluster point. This notion of sequential compactness is classically equivalent to the usual classical one: this follows from the observations that, classically, a given sequence in X either has a cluster point or it does not have one, and that in the latter case, it has at most one cluster point in our sense. The classical notion of sequential compactness is totally useless constructively: if every sequence in the discrete two–point space $\{0, 1\}$ had a convergent subsequence, then the standard equality relation on $2^{\mathbb{N}}$ would be decidable.[3] However, $\{0, 1\}$ is sequentially compact in our constructive sense: if $(x_n)_{n \geqslant 0}$ is a sequence in $\{0, 1\}$ with at most one cluster point, then we can compute $\delta > 0$ and N such that either $|x_n| \geqslant \delta$ for all $n \geqslant N$ or else $|1 - x_n| \geqslant \delta$ for all $n \geqslant N$; in the first case, $x_n = 1$ for all $n \geqslant N$, and in the second, $x_n = 0$ for all $n \geqslant N$. Is this a special case, or is every compact—that is, complete and totally bounded—metric space sequentially compact in the constructive sense? To see that this question fits in with our general scheme, take $P(x)$ to be the statement

$$\forall_{\varepsilon>0} \forall_n \exists_{m>n} \left(\rho(x, x_m) < \varepsilon\right),$$

and $P'(x)$ to be the statement

$$\exists_{\delta>0} \exists_N \forall_{n \geqslant N} \left(\rho(x, x_n) \geqslant \delta\right).$$

Note that it is definitely false in recursive constructive mathematics—BISH plus the Church–Markov–Turing thesis—that every sequence in $[0, 1]$ with at most one cluster point has a cluster point: just consider a Specker sequence ([6], Chapter 3). Since recursive constructive mathematics is a model of BISH, there is no possibility of proving within BISH alone that the compact space $[0, 1]$ is sequentially compact. In fact, as one of our theorems shows, we need some version of the fan theorem to prove even a weak form of sequential compactness for complete, totally bounded metric spaces. This is an appropriate place for us to recall

[2] This notion of 'at most one cluster point' is weaker than the one used in [7]. Our notion of 'sequentially compact' later in the present paper also differs from the one with the same name in [7].

[3] In fact, the decidability of the equality on $2^{\mathbb{N}}$ is equivalent to every complete, totally bounded metric space being sequentially compact in the classical sense; see [5, 10].

some facts about fans and bars. We write $\mathbf{x}, \mathbf{y}, \ldots$ for the elements $(x_n)_{n \geqslant 0}, (y_n)_{n \geqslant 0}, \ldots$ of the **complete binary fan** $2^{\mathbb{N}}$; and for each $\mathbf{x} \in 2^{\mathbb{N}}$ and each natural number n we write

$$\overline{\mathbf{x}}(n) = (x_0, \ldots, x_{n-1}),$$

where it is understood that $\overline{\mathbf{x}}(0)$ is the empty sequence. The space $2^{\mathbb{N}}$ is compact relative to the metric ρ defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \inf\left\{2^{-n} : \overline{\mathbf{x}}(n) = \overline{\mathbf{y}}(n)\right\}$$

(see [6], Chapter 5). A set B of finite binary sequences is said to be

- ▷ **detachable** if
$$\forall_{\mathbf{x} \in X} \forall_n (\overline{\mathbf{x}}(n) \in B \vee \overline{\mathbf{x}}(n) \notin B);$$
- ▷ a **bar** for $2^{\mathbb{N}}$ if
$$\forall_{\mathbf{x} \in X} \exists_{n \geqslant 1} (\overline{\mathbf{x}}(n) \in B);$$
- ▷ a **uniform bar** for $2^{\mathbb{N}}$ if
$$\exists_N \forall_{\mathbf{x} \in X} \exists_{n \leqslant N} (\overline{\mathbf{x}}(n) \in B).$$

Brouwer's fan theorem for detachable bars ($\mathbf{FT}_D$) says that *every detachable bar for* $2^{\mathbb{N}}$ *is a uniform bar*; see [6] (Chapter 5) or [8] for more details. Let X be a metric space. We say that a sequence $(x_n)_{n \geqslant 0}$ in X has **continuously at most one cluster point**, or is a $\mathbf{C_1CP}$ **sequence**, if there exists a uniformly continuous function $g : X \times X \to \mathbb{R}$ such that for all distinct points $\mathbf{x}, \mathbf{y}$ of X, $g(\mathbf{x}, \mathbf{y}) > 0$ and

$$\exists_{n \geqslant 1} (\forall_{n \geqslant N} (\rho(\mathbf{x}, \mathbf{x}_n) \geqslant g(\mathbf{x}, \mathbf{y})) \vee \forall_{n \geqslant N} (\rho(\mathbf{y}, \mathbf{x}_n) \geqslant g(\mathbf{x}, \mathbf{y}))).$$

A weak form of sequential compactness would be the property that every $\mathrm{C_1CP}$ sequence has a cluster point. Our next three results reveal that $\mathbf{FT}_D$, added to BISH, is both necessary and sufficient for us to be able to prove that every $\mathrm{C_1CP}$ sequence in a compact metric space has a cluster point.

Theorem 0.1
BISH ⊢ Let B be a detachable bar for $2^{\mathbb{N}}$. Then there exist a sequence $(\mathbf{x}_n)_{n \geqslant 0}$ in $2^{\mathbb{N}}$, and a uniformly continuous mapping $g : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{R}$, with the following properties.

(i) For each pair $(\mathbf{x}, \mathbf{y})$ of distinct points of $2^{\mathbb{N}}$, $g(\mathbf{x}, \mathbf{y}) > 0$ and there exists a positive integer N such that

$$\forall_{n \geqslant N} (\rho(\mathbf{x}, \mathbf{x}_n) \geqslant g(\mathbf{x}, \mathbf{y})) \vee \forall_{n \geqslant N} (\rho(\mathbf{y}, \mathbf{x}_n) \geqslant g(\mathbf{x}, \mathbf{y})).$$

(ii) If $(\mathbf{x}_n)$ has a cluster point, then B is a uniform bar for $2^{\mathbb{N}}$.

Proof. Since B is detachable, for each $\mathbf{x} \in 2^{\mathbb{N}}$ the natural number

$$\sigma(\mathbf{x}) = \inf\{n \geqslant 0 : \overline{\mathbf{x}}(n) \in B\}$$

is well defined. Construct an increasing binary sequence $(\lambda_n)_{n \geqslant 0}$ such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad \exists_{\mathbf{x} \in X} (\sigma(\mathbf{x}) > n), \\ \lambda_n = 1 \quad &\Rightarrow \quad \forall_{\mathbf{x} \in X} (\sigma(\mathbf{x}) \leqslant n). \end{aligned}$$

We may assume that $\lambda_0 = 0$. If $\lambda_n = 0$, choose $\mathbf{x}_n \in 2^{\mathbb{N}}$ with $\sigma(\mathbf{x}_n) > n$. If $\lambda_n = 1$, set $\mathbf{x}_n = \mathbf{x}_{n-1}$. Then $(\mathbf{x}_n)_{n\geqslant 0}$ is a sequence in $2^{\mathbb{N}}$. To prove that it has continuously at most one cluster point, let $\mathbf{x}, \mathbf{y}$ be distinct points of $2^{\mathbb{N}}$, and let $\nu = \sigma(\mathbf{x}) + 1$. If $\lambda_\nu = 1$, then there exists $N \leqslant \nu$ such that $\lambda_N = 1 - \lambda_{N-1}$, and therefore $\bar{\mathbf{x}}_n = \bar{\mathbf{x}}_{N-1}$ for all $n \geqslant N$. Either $\rho(\mathbf{x}_{N-1}, \mathbf{x}) > \frac{1}{2}\rho(\mathbf{x}, \mathbf{y})$ or else $\rho(\mathbf{x}_{N-1}, \mathbf{y}) > \frac{1}{2}\rho(\mathbf{x}, \mathbf{y})$. In the first case, $(\mathbf{x}_n)_{n\geqslant 0}$ is eventually bounded away from $\mathbf{x}$; in the second, it is eventually bounded away from $\mathbf{y}$. Thus we may assume that $\lambda_\nu = 0$. For each $n \geqslant \nu$ with $\lambda_n = 0$ we have $\sigma(\mathbf{x}_n) > n \geqslant \nu > \sigma(\mathbf{x})$. It follows from this and the definition of the function σ that there exists $j < \sigma(\mathbf{x})$ such that $\mathbf{x}_n(j) \neq \mathbf{x}(j)$; whence $\rho(\mathbf{x}_n, \mathbf{x}) \geqslant 2^{-j} \geqslant 2^{-\sigma(\mathbf{x})}$. On the other hand, if $n \geqslant \nu$ and $\lambda_n = 1$, then there exists k such that $\nu + 1 \leqslant k < n$, $\lambda_k = 1 - \lambda_{k-1}$, and $\mathbf{x}_n = \mathbf{x}_{k-1}$. Then

$$\sigma(\mathbf{x}_n) = \sigma(\mathbf{x}_{k-1}) > k - 1 \geqslant \nu > \sigma(\mathbf{x}),$$

so, by the same argument used before, $\rho(\mathbf{x}_n, \mathbf{x}) \geqslant 2^{-\sigma(\mathbf{x})}$. Thus in all cases we have

$$\forall_{n\geqslant\sigma(\mathbf{x})} (\rho(\mathbf{x}_n, x) \geqslant g(\mathbf{x}, \mathbf{y})) \vee \forall_{n\geqslant\sigma(\mathbf{x})} (\rho(\mathbf{x}_n, \mathbf{y}) \geqslant g(\mathbf{x}, \mathbf{y})),$$

where the uniformly continuous function $g : X \times X \to \mathbb{R}$ is defined by

$$g(\mathbf{x}, \mathbf{y}) = \min\left\{2^{-\sigma(\mathbf{x})}, \frac{1}{2}\rho(\mathbf{x}, \mathbf{y})\right\}.$$

This completes the proof that $(\mathbf{x}_n)_{n\geqslant 0}$ has at most one cluster point. Now suppose that it has a cluster point ξ, and choose $n > \sigma(\xi)$ such that $\rho(\xi, \mathbf{x}_n) < 2^{-\sigma(\xi)}$. Then $\bar{\xi}(\sigma(\xi)) = \bar{\mathbf{x}}_n(\sigma(\xi))$, so $\sigma(\mathbf{x}_n) = \sigma(\xi)$. But if $\lambda_n = 0$, then $\sigma(\mathbf{x}_n) > n > \sigma(\xi)$, a contradiction. Hence $\lambda_n = 1$, and therefore $\sigma(\mathbf{x}) \leqslant n$ for all $\mathbf{x} \in 2^{\mathbb{N}}$. In other words, B is a uniform bar. ∎

COROLLARY 0.2
BISH $\vdash$ If every C_1CP sequence in $2^{\mathbb{N}}$ has a cluster point, then $\mathbf{FT}_D$ holds.

PROOF. Immediate consequence of Theorem 0.1. ∎

Conversely, we have

THEOREM 0.3
BISH + FT_D $\vdash$ In a compact metric space every C_1CP sequence converges.

PROOF. Let (X, ρ) be a compact metric space, and let $(x_n)_{n\geqslant 1}$ be a C_1CP sequence in X. There exists a uniformly continuous map $g : X \times X \to \mathbb{R}$ such that for all distinct points x, y of X, $g(x, y) > 0$ and

$$\exists_N (\forall_{n\geqslant N} (\rho(x, x_n) > g(x, y)) \vee \forall_{n\geqslant N} (\rho(y, x_n) > g(x, y))).$$

Fix $\varepsilon > 0$ such that

$$K_\varepsilon = \{(x, y) \in X \times X : \rho(x, y) \geqslant \varepsilon\}$$

is compact ([4], page 98, Theorem (4.9)). It follows from $\mathbf{FT}_D$ that

$$0 < \delta = \inf\{g(x, y) : x, y \in K_\varepsilon\}$$

(see [6], pages 127–128). Construct a finite $\delta/2$–approximation

$$\{(\xi_1, \eta_1), \ldots, (\xi_m, \eta_m)\}$$

to K_ε. For each k $(1 \leqslant k \leqslant m)$ there exists a positive integer N_k such that

$$\forall_{n \geqslant N_k} \left(\rho\left(\xi_k, x_n\right) > g(\xi_k, \eta_k)\right) \vee \forall_{n \geqslant N_k} \left(\rho\left(\eta_k, x_n\right) > g\left(\xi_k, \eta_k\right)\right)$$

and therefore

$$\forall_{n \geqslant N_k} \left(\rho\left(\xi_k, x_n\right) \geqslant \delta\right) \vee \forall_{n \geqslant N_k} \left(\rho\left(\eta_k, x_n\right) \geqslant \delta\right). \tag{0.3}$$

Define

$$N = \max\left\{N_1, \ldots, N_m\right\}$$

and consider any $(x, y) \in K_\varepsilon$. There exists k $(1 \leqslant k \leqslant m)$ such that

$$\max\left\{\rho\left(x, \xi_k\right), \rho\left(y, \eta_k\right)\right\} = \rho\left(\left(x, y\right), \left(\xi_k, \eta_k\right)\right) < \frac{\delta}{2}.$$

If, for example, the first alternative in (0.3) holds, then for each $n \geqslant N$ we have $n \geqslant N_k$ and therefore

$$\rho\left(x, x_n\right) \geqslant \rho\left(\xi_k, x_n\right) - \rho\left(x, \xi_k\right) \geqslant \frac{\delta}{2}.$$

Thus we have found $\delta > 0$ and a positive integer N such that

$$\forall_{(x,y) \in K_\varepsilon} \left(\forall_{n \geqslant N} \left(\rho\left(x, x_n\right) \geqslant \frac{\delta}{2}\right) \vee \forall_{n \geqslant N} \left(\rho\left(y, x_n\right) \geqslant \frac{\delta}{2}\right)\right).$$

It follows that if $n \geqslant N$, then $\rho\left(x_n, x_N\right) \leqslant \varepsilon$: for if $\rho\left(x_n, x_N\right) > \varepsilon$, then $\left(x_n, x_N\right) \in K_\varepsilon$, so either $\rho\left(x_n, x_n\right) > \delta/2$ or $\rho\left(x_N, x_N\right) > \delta/2$, both of which alternatives are absurd. Hence $\rho\left(x_i, x_j\right) \leqslant 2\varepsilon$ for all $i, j \geqslant N$; so $\left(x_n\right)_{n \geqslant 1}$ is a Cauchy sequence, which, since X is complete, converges to a limit in X. ∎

For an application of the foregoing theorem we prove a folklore result (cf. [4] page 63, Problem 12).

Lemma 0.4
Let f be a mapping of a compact metric space X into a metric space Y. Then there exists a uniformly continuous mapping $\delta : (0, 1] \to \mathbb{R}^+$ such that for each $\varepsilon \in (0, 1]$, if $x, y \in X$ and $\rho\left(x, y\right) \leqslant \delta\left(\varepsilon\right)$, then $\rho\left(f(x), f(y)\right) \leqslant \varepsilon$.

Proof. Using the principle of countable choice, construct a strictly decreasing sequence $\left(\delta_n\right)_{n \geqslant 1}$ of positive numbers converging to 0, such that for each n, if $x, y \in X$ and $\rho\left(x, y\right) \leqslant \delta_n$, then $\rho\left(f(x), f(y)\right) \leqslant 1/n$. Define the function δ on the set

$$D = \bigcup_{n \geqslant 1} \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

by

$$\delta\left(t\frac{1}{n+1} + (1-t)\frac{1}{n}\right) = t\delta_{n+2} + (1-t)\,\delta_{n+1} \quad (n \geqslant 1,\ 0 < t < 1).$$

Then δ is uniformly continuous on D, and so extends to a uniformly continuous mapping $\delta : (0, 1] \to \mathbb{R}^+$. Let $0 < \varepsilon < 1$, and consider points $x, y \in X$ with $\rho\left(x, y\right) \leqslant \delta(\varepsilon)$. Suppose that $\rho(f(x), f(y)) > \varepsilon$. If $\varepsilon \in D$, then there exists a unique positive integer n such that $\frac{1}{n+1} < \varepsilon < \frac{1}{n}$; since $\delta(\varepsilon) < \delta_{n+1}$, this yields

$$\rho(f(x), f(y)) \leqslant \frac{1}{n+1} < \varepsilon,$$

a contradiction. Hence $\varepsilon \notin D$. In that case, choosing n such that $\frac{1}{n+2} < \varepsilon < \frac{1}{n}$, we see that $\varepsilon = \frac{1}{n+1}$; whence $\delta(\varepsilon) = \delta_{n+2}$ and therefore $\rho(f(x), f(y)) \leqslant 1/(n+2) < \varepsilon$, again a contradiction. We conclude that $\rho(f(x), f(y)) \leqslant \varepsilon$ after all. ∎

Here is our application of Theorem 0.3.

THEOREM 0.5
$\mathbf{FT}_D \vdash$ Let X be a compact metric space, f a uniformly continuous mapping of X into $\mathbb{R}$, and $M = \sup f$. Suppose that f has **at most one maximum**, in the sense that

$$\forall_{x,y} \in X\,(x \neq y \Rightarrow (f(x) < M) \vee (f(y) < M)).$$

Then there exists $x \in X$ such that $f(x) = M$.

PROOF. By Lemma 0.4, there exists a uniformly continuous mapping $\delta : (0,1] \to \mathbb{R}^+$ such that if $0 < \varepsilon < 1$, if $x, y \in X$, and if $\rho(x,y) \leqslant \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$. Construct a sequence $(x_n)_{n \geqslant 1}$ in X such that $f(x_n) > M - \frac{1}{n}$ for each n. Given distinct points x, y of X, let

$$\varepsilon = \frac{1}{5}\max\{M - f(x), M - f(y)\},$$

which is positive by our hypothesis of 'at most one maximum'. Choose a positive integer $N > 1/\varepsilon$. Either $M - f(x) > 2\varepsilon$ or $M - f(y) > 2\varepsilon$. In the first case, for all $n \geqslant N$ we have

$$f(x_n) > M - \frac{1}{n} \geqslant M - \frac{1}{N} > M - \varepsilon > f(x) + \varepsilon.$$

Handling the second case similarly, we now see that

$$\forall_{n \geqslant N}(f(x_n) > f(x) + \varepsilon) \vee \forall_{n \geqslant N}(f(y) > f(x) + \varepsilon).$$

Hence

$$\forall_{n \geqslant N}(|x_n - x| \geqslant \delta(\varepsilon)) \vee \forall_{n \geqslant N}(|x_n - y| \geqslant \delta(\varepsilon)).$$

It follows that

$$g(x,y) = \delta\left(\frac{1}{5}\max\{M - f(x), M - f(y)\}\right)$$

defines a function with the properties required for an application of Theorem 0.3 to the sequence (x_n). Since we are assuming $\mathbf{FT}_D$, it follows from that theorem that this sequence converges to a limit $x \in X$. The continuity of f ensures that $f(x) = M$. ∎

Theorem 0.5 was proved by a different method in [3]. However, the foregoing illustration of the application of Theorem 0.3 is not without interest, as it is clearly an analogue of the elementary sequential–compactness–based classical proof that f has a maximum. In fact, our proof of Theorem 0.5 actually shows that if every C_1CP sequence in a compact metric space X converges, then every uniformly continuous map from X to $\mathbb{R}$ with at most one maximum actually has a maximum. We can then deduce Theorem 0.1 as a corollary of Theorem 5 of [3]. However, the proof of Theorem 0.1 given above is direct, without any detour into considerations of continuous functions with at most one maximum. With the full form of **Brouwer's fan theorem**—that is, $\mathbf{FT}_D$ without the detachability hypothesis—we can prove, as follows, that every complete, totally bounded metric space X is sequentially compact. Let $(x_n)_{n\geqslant 1}$ be a sequence in X that has at most one cluster point. Choose an arbitrarily small $\varepsilon > 0$ such that the set K_ε (as defined earlier) is compact. For each $(x,y) \in K_\varepsilon$ there exist $\delta > 0$ and a positive integer N such that (0.2) holds. Brouwer's fan theorem implies that K_ε has the Heine–Borel covering property of K_ε; so there exist finitely many points $(\xi_1,\eta_1),\ldots,(\xi_m,\eta_m)$ of K_ε, positive numbers $\delta_1,\ldots,\delta_m$, and positive integers $N_1,\ldots,N_m$ such that

- the sets
$$\left\{(x,y) : \rho\left((x,y),(\xi_i,\eta_i)\right) < \frac{\delta_i}{2}\right\} \quad (1 \leqslant i \leqslant m)$$
cover K_ε, and
- for each i $(1 \leqslant i \leqslant m)$,
$$\forall_{n\geqslant N_i}\left(\rho(\xi_i,x_n) \geqslant \delta_i\right) \vee \forall_{n\geqslant N_i}\left(\rho(\eta_i,x_n) \geqslant \delta_i\right).$$

Let
$$\delta = \min_{1\leqslant i\leqslant m} \frac{\delta_i}{2}, \qquad N = \max_{1\leqslant i\leqslant m} N_i.$$

Given $(x,y) \in K_\varepsilon$, we can find i such that
$$\max\left\{\rho(x,\xi_i),\rho(y,\eta_i)\right\} = \rho\left((x,y),(\xi_i,\eta_i)\right) < \frac{\delta_i}{2}.$$

If $\rho(\xi_i,x_n) \geqslant \delta_i$ for all $n \geqslant N_i$, then for all $n \geqslant N$ we have
$$\begin{aligned} \rho(x,x_n) &\geqslant \rho(\xi_i,x_n) - \rho(\xi_i,x) \\ &\geqslant \delta_i - \frac{\delta_i}{2} = \frac{\delta_i}{2} \geqslant \delta. \end{aligned}$$

Likewise, if $\rho(\eta_i,x_n) \geqslant \delta_i$ for all $n \geqslant N_i$, then $\rho(y,x_n) \geqslant \delta$ for all $n \geqslant N$. Thus we have found $\delta > 0$ and a positive integer N such that (0.2) holds for all $(x,y) \in K_\varepsilon$. Arguing as at the end of the proof of Theorem 0.3, we can now show that (x_n) is a Cauchy sequence, and hence converges, in X. Finally, we have an unanswered question: does $\mathbf{FT}_D$ imply that $2^{\mathbb{N}}$, and hence every complete, totally bounded metric space (see [15], Vol. 2, page 364), is sequentially compact? In other words, can we weaken the hypotheses of Theorem 0.3 by replacing the existence of the function g by the simpler condition that the sequence $(x_n)_{n\geqslant 1}$ have at most one cluster point?

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